The influence of measurement noise on the performance of a discrete Linear Quadratic Gaussian controller for adaptive optics systems

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Abstract—Adaptive optics systems represent a significant challenge for the controller design. The output disturbance must be compensated despite a significant measurement noise and time-delays. This paper presents the formulation of a discrete Linear Quadratic Gaussian controller, which includes a realistic model of the measurement noise. The correspondence between the photosensor noise in the sensor and the measurement noise is established. Different cases of the system’s operation are considered. The influence of the measurement noise on the discrete LQG controller in adaptive optics systems is discussed.

Keywords - optimal control, adaptive optics, measurement noise, numerical simulations.

I. INTRODUCTION

The turbulence of the Earth’s atmosphere introduces inhomogeneity in the air refractive index. Such an inhomogeneity leads to blurring of astronomical images acquired by ground-based telescopes. Adaptive optics can compensate the turbulence in real-time using wavefront sensors (WFS) and deformable mirrors (DM). The closed-loop operation is performed by a controller that generates control commands for the DM using noisy and delayed WFS measurements.

The usual choice for the control is a simple PI controller [1], where the gain can be adjusted for each mode [2]. More advanced controllers such as LQG have been designed and simulated for modal wavefront correction [3], both for classical [4] and multi-conjugate adaptive optics. The experimental demonstration of the LQG controllers in adaptive optics was provided in [5]. The online parameter adjustment for LQG controller was also proposed [6]. The experimental validation of optimal control was shown [5], [7] to outperform the classic integrator based control.

The LQG controllers for the adaptive optics systems are often designed in simplified (no actuators dynamics) and unrealistic (no measurement noise) assumptions [8]. This short contribution addresses the problem of adequate measurement noise modelling. The correspondence between the photosensor noise in the WFS and the measurement noise is difficult to elaborate analytically. We use a comprehensive model of the CCD photosensor [9] to obtain the corresponding noise value for various cases of photosensor ADC quantisation, signal and noise levels. The obtained measurement noise levels are used for the performance analysis of the formulated LQG controller.

II. THE DESCRIPTION OF THE PROBLEM

From the control standpoint, the adaptive optics system is a Linear Time Invariant (LTI) multi-input-multi-output (MIMO) system that consists of a sensor (wavefront sensor), a controller, and a corrector (deformable mirror) for the real-time sensing and compensating of an output disturbance (atmospheric turbulence). A typical scheme is shown in Fig. 1.

![Fig. 1. The basic control scheme of an Adaptive Optics system.](image-url)

The system is a subject to both an output disturbance (atmospheric turbulence that must be corrected) and measurement noise from the CCD or CMOS photosensor inside the WFS. The aim of the controller is to track zero reference (a flat wavefront) in the presence of output disturbances and a significant measurement noise.
A. The model of the output disturbance

The output disturbance in this system is due to atmospheric turbulence, which is essentially a wavefront phase distortion. The wavefront can be thought as a surface of all light rays coming from a distant star. If there is no atmospheric turbulence, the wavefront surface will be flat, which means that all the rays have passed the same distance. The atmospheric turbulence produces inhomogeneity in the air refractive index that leads to different path lengths for different light rays. Hence the wavefront will be non-flat, which causes the blur on the astronomical images. The goal of the controller in the adaptive optics system is to counteract the non-flat (turbulent) wavefront by bending the DM in the opposite direction.

The atmospheric turbulence can be represented by a discrete-time state space model driven by a white noise, as seen in Fig. 2. A state space model of the atmospheric turbulence (output disturbance) is given by:

\[ x_{\text{atm}}[k + 1] = A_{\text{atm}}x_{\text{atm}}[k] + B_{\text{atm}}\xi_{\text{atm}}[k] \]
\[ y_{\text{atm}}[k] = C_{\text{atm}}x_{\text{atm}}[k] \]

where \( x_{\text{atm}}[k] \) is the disturbance state and \( \xi_{\text{atm}}[k] \) is a zero mean, unit variance white noise process.

![Atmospheric turbulence state space model.](Image)

Fig. 2. Atmospheric turbulence state space model.

It was shown [10] that the autoregressive AR(1) model provides limited performance due to the nature of the approximate model. On the other hand, an AR(5) model leads to a loss of performance due to increased sensitivity to variations of temporal characteristics of the atmospheric turbulence. Nonetheless, the AR(1) model provides a reasonable agreement with the Kolmogorov model [11] in good seeing conditions and is used in this paper.

B. The model of the Wavefront Sensor

The dynamics of the wavefront sensor (WFS) are characterized by the integration time \( T_f \) of the photosensor. The output of the sensor is the average of the signal from \( t \) to \( t + T_f \). The delay due to data processing adds the time delay \( \tau_{\text{wfs}} \). Therefore, the transfer function of the WFS can be represented in Laplace domain as:

\[ G_{\text{WFS}}(z) = \frac{1 - e^{-T_f z}}{T_f s} \cdot e^{\tau_{\text{wfs}}} \]

Equation 3 is schematically presented in Fig. 3. The discrete data of the wavefront are contaminated by measurement noise from the WFS, which is described in detail in the Subsection V-A. The measurement noise is represented as an additive white Gaussian noise; Subsection V-C provides an explanation of why this is a valid choice for the simulations.

![The model of wavefront sensor (WFS).](Image)

Fig. 3. The model of wavefront sensor (WFS).

C. The model of the Deformable mirror

The dynamics of the continuous-facesheet deformable mirror (DM) with piezoelectric actuators can be divided into the dynamics of a surface and the dynamics of the actuators. In this paper, only the dynamics of the actuators is considered; the coupling between actuators and the surface dynamics is neglected. The dynamics of the actuators [12] in a DM is:

\[ G_{\text{DM}}(s) = \frac{1}{s^2/a + 1} \]

where the parameter \( a \) depends on the specific equipment used in the adaptive optics system.

D. The overall model of the Adaptive Optics system

The model of the system consists of WFS dynamics, actuators of the DM, and a non-integer time delay of the controller.

In order to account for the non-integer time delay, we follow the reference [13] and divide the delay into an integer part \( n \) and fractional part \( m \in [0, 1) \). Therefore, the complete system delay \( \tau_d \) will be represented as: \( \tau_d = (n - m) \cdot T_f \). Because \( n \) is an integer, the term \( \exp[-nT_f s] \) will be transformed in the Z-domain as \( z^{-n} \). The residual fractional delay \( \exp[mT_f s] \) will be absorbed into the coefficients of the discretized model [13].

The discretized equivalent of the continuous system can be represented as a transfer function:

\[ G_{\text{sys}}(z) = \frac{z^\beta + \alpha \beta}{z^n + 2 \cdot z^{n+1} + e^{-\alpha T_f}}, \]
\[ \alpha = \frac{e^{-amT_f} - e^{-at_f}}{1 - e^{-amT_f}}, \quad \beta = 1 - e^{-amT_f} \]

that can be converted into state space form:

\[ x_{\text{sys}}[k + 1] = A_{\text{sys}}x_{\text{sys}}[k] + B_{\text{sys}}\xi_{\text{sys}}[k] \]
\[ y_{\text{sys}}[k] = C_{\text{sys}}x_{\text{sys}}[k] \]

The state space formulation of the output disturbance is used for the augmented state matrices of the complete system.
The system is assumed to be LTI with state $x[k]$, input $u[k]$, and measurement $y[k]$ given by:

$$\begin{align*}
x[k+1] &= Ax[k] + Bu[k] + G\xi[k] \\
y[k] &= Cx[k] + Du[k] + \eta[k]
\end{align*}$$

(9) (10)

here $\xi[k]$ is the process noise that is zero-mean Gaussian white noise and $\eta[k]$ is the measurement noise that is zero-mean Gaussian white noise. The matrices $A$, $B$, and $C$ in the Eq. 9 must be augmented in order to account for both the plant dynamics and the output disturbance:

$$A = \begin{bmatrix} A_{\text{sys}} & 0 \\ 0 & A_{\text{atm}} \end{bmatrix} \quad B = \begin{bmatrix} B_{\text{sys}} \\ 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ B_{\text{atm}} \end{bmatrix}$$

(11)

$$C = [C_{\text{sys}}, C_{\text{atm}}] \quad D = 0$$

(12)

In order to derive a performance criterion for the controller, one can use the Strehl ratio\(^1\). The residual wavefront phase (uncompensated disturbance) reduces the Strehl ratio, which can be expressed using the Maréchal approximation [14]:

$$S_{\text{Maréchal}} \approx 1 - 2\pi^2 \left( \frac{\Delta \phi_r}{\lambda} \right)^2$$

(13)

where $\lambda$ is the observation light’s wavelength (m) and $\Delta \phi_r$ is the residual wavefront phase (m).

The objective of the controller is to minimise the wavefront phase variance (i.e., keep the wavefront flat), which is equivalent to maximisation of the Strehl ratio. The cost function for the infinite-horizon problem can be formulated as:

$$J = \sum_{k=0}^{\infty} \left( x^T[k]Qx[k] + u^T[k]Ru[k] \right)$$

(14)

The matrix $Q$ can be chosen as $Q = C^T C$. Such a choice, however, causes numerical problems in the solution of the Riccati equations (see Subsection III-D for details). The matrix $R$ is set $R = 0$ since we do not penalise the control effort.

### A. State feedback gain evaluation

The feedback gain can be found using the solution of the discrete-time algebraic Riccati equations (DARE):

$$A^T\Omega_K - A^T\Omega_K B(B^T\Omega_K B + R)^{-1}B^T\Omega_K A + Q = \Omega_K$$

(15)

The solution of DARE in Eq. 15 is the matrix $\Omega_K$, which is used for the state feedback gain calculation:

$$K_{\text{gain}} = (B^T\Omega_K B + R)^{-1}B^T\Omega_K A$$

(16)

where $Q = C^T C = [C_{\text{sys}}, C_{\text{atm}}]^T[C_{\text{sys}}, C_{\text{atm}}]$ and $R = 0$.

### B. State observer gain evaluation

The observer gain is found from the dual DARE:

$$A\Omega_L A^T - A\Omega_L C^T(C\Omega_L C^T + \Theta)^{-1}C\Omega_L A^T + V = \Omega_L$$

(17)

where $V = GG^T$ and $G = [0,B_{\text{atm}}]^T$ as indicated in the Eq. 11. The choice of $V = GG^T$ can lead to the same type of the numerical problems as the case of the matrix $Q$ in state feedback case (this issue is discussed in Subsection III-D).

Using the solution $\Omega_L$ of DARE in Eq. 17, we can evaluate the observer gain $L_{\text{gain}}$:

$$L_{\text{gain}} = A\Omega_L C^T(C\Omega_L C^T + W)^{-1}$$

(18)

where $W$ is the measurement noise covariance matrix. The feedthrough gain $L_{\text{ft}}$ can be found in the similar way.

### C. The formulation of the Discrete LQG controller

By combining the solutions for the feedback and the observer from the previous subsections, we can formulate the controller as:

$$u[k] = -K_{\text{gain}} \left( \hat{x}[k] + L_{\text{ft}}(y[k] - C\hat{x}[k]) \right)$$

(19)

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L_{\text{gain}}(y[k] - C\hat{x}[k]).$$

(20)

More convenient form for the implementation is:

$$u[k] = -\Psi \hat{x}[k] + L_{\text{ft}}y[k]$$

(21)

$$\hat{x}[k+1] = \Psi \hat{x}[k] + \Gamma y[k],$$

(22)

where:

$$\Psi = K_{\text{gain}} + L_{\text{ft}} C$$

(23)

$$\Gamma = L_{\text{gain}} + BL_{\text{ft}}$$

(24)

Then the controller can be converted to the transfer function representation, which is easier for practical implementation:

$$C(z) = \Gamma (zI - \Psi)^{-1} \cdot \Gamma + L_{\text{ft}}$$

(26)

In these numerical experiments, the controller is used in the form of Eq. 26 as shown in the control scheme Fig. 1.

### D. Numerical stability and regularisation of the Riccati equations

The matrix $Q = [C_{\text{sys}}, C_{\text{atm}}]^T[C_{\text{sys}}, C_{\text{atm}}]$ that is used in the cost function in Eq. 14 may lose the positive semi-definiteness and become ill-conditioned. That in turn leads to numerical problems in solution of Riccati equations.

\(^1\)The Strehl ratio is the ratio of the peak intensity of a measured point spread function (PSF) to the peak intensity of a perfect diffraction-limited PSF for the same optical system. Strehl ratio values lying within the ambit of $[S \in 0, 1]$ and the closer $S$ to 1 the better (sharper) the image.
We use the diagonal loading technique for the regularisation that is analogous to Tikhonov regularisation [15]:

$$Q = Q + \delta I,$$

(27)

where $\delta$ is the regularisation term $\delta = \epsilon / T_f$, $T_f$ is the sampling period, $\epsilon$ a floating-point relative accuracy ($\epsilon \approx 2.2 \cdot 10^{-16}$ in our case), and $I$ is an identity matrix. By increasing the regularisation term in Eq. 27 up to $10^7 \delta I$, we have encountered that the conditioning number $\kappa(Q)$ is improved up to $10^5$. However, by that point the calculated LQG controller becomes unstable. Reasonable values of the regularisation term $\delta$ are found to be within the ambit $\epsilon \ldots 10^3 \epsilon$.

IV. INITIAL PARAMETERS OF THE NUMERICAL SIMULATIONS

The goals of the numerical simulations are 1) to establish the correspondence between the noise in the WFS and the measurement noise and 2) evaluate the influence of measurement noise on the performance of the discrete LQG controller. The case of bright, medium, and dim guide star corresponds to high, medium, and low level of a photosensor signal. The variance of the measurement noise is estimated using the optical simulator.

A. Output disturbance (atmospheric turbulence)

The atmospheric disturbance was simulated as an AR(1) process. This model describes the atmospheric turbulence with Kolmogorov-type power spectrum. A white noise process with the standard deviation of $500 \cdot 10^{-9}$ (m) was used as an input for the output disturbance. The model of the disturbance is discrete: the sampling rate in the Band-Limited White Noise block is $T_f / 2$ (i.e., two samples per one sample of the WFS).

B. Parameters of the simulated plant

The model of the Shack-Hartmann WFS with $32 \times 32$ lenslets was used for the numerical simulations. The observation wavelength was $\lambda = 500$ nm, the size of the lenslet was $85 \mu m$ (with $17 \times 17$ pixels per lenslet, each pixel is $5 \mu m$), focal distance of the lenslet was $10$ mm. The weighted centre of gravity algorithm was used for the centroiding. The sampling period of the WFS was $T_f = 10^{-3}$ sec. The time delay introduced by the WFS was equivalent to $1$ frame. Such parameters can be considered typical for the adaptive optics systems.

The DM was represented by the dynamics of the actuators according to Eq. 4:

$$G_{DM}(s) = \frac{1}{s / \alpha + 1}$$

(28)

where the parameter $\alpha$ was $2 \cdot 10^{-3}$. In the current simulations, the SISO output case of zonal control is considered and the coupling between the actuators in the DM is neglected.

2Consider the singular value decomposition (SVD) of the matrix $Q$ as $Q = U \Sigma V^H$, where $\Sigma$ is the diagonal matrix of singular values $\sigma_i$. Define the condition number [16] as $\kappa(Q) = \sigma_{max} / \sigma_{min}$.

The discrete LQG controller was formulated as described in Section III. The controller was assumed to have a delay$^3$ equal to $0.9$ of the sampling time, that is $\tau_d = (n - m) \cdot T_f$, where $n = 1$ and $m = 0.1$ (see Subsection II-D).

C. Parameters for the overall model of the Adaptive Optics system

Using the described augmentation of the LQG controller from Section III, the resulting discrete state space system is the following:

$$A = \begin{bmatrix} A_{sys} & 0 \\ 0 & A_{atm} \end{bmatrix} = \begin{bmatrix} 0.1353 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} B_{sys} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} C_{sys} \\ C_{atm} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.1813 & 0.6834 & 1.0000 \end{bmatrix}$$

$D = 0$

The system has been converted to transfer function $C(z)$ as in Eq. 26 and used for the numerical simulations.

V. RESULTS ON DERIVATION OF THE EQUIVALENT MEASUREMENT NOISE FROM THE PHOTOSENSOR NOISE

The noise is added by the photosensor to the intermediate data that are used for the reconstruction of the signal. The noise is signal-dependent, which means that the amount of noise will vary depending on the ADC and the signal level.

A. The relationship between the photosensor noise and the measurement noise

The correspondence of the photosensor noise and the measurement noise is difficult to elaborate analytically. It is because the photosensor noise is added to the intensity image $I$, which is produced by the lenslets of the Shack-Hartman WFS. The intensity image $I$ is used for the calculation of centroid coordinates $(x_k, y_k)$:

$$x_k + N_1 = \frac{\sum_{i,j} x_{i,j} (I_{i,j} + N)}{\sum_{i,j} (I_{i,j} + N)}$$

(29)

where $N$ is the photosensor noise; $N_1$ is the centroids noise due to the noise in the intensity image $I$ that is used for the calculation of turbulent $(x_{ck}, y_{ck})$ and reference $(x_{rk}, y_{rk})$ centroid coordinates. The wavefront angles can be calculated:

$$\beta_x \approx (x_{rk} - x_{ck}) l_{pix} / f, \quad \beta_y \approx (y_{rk} - y_{ck}) l_{pix} / f,$$

(30)

$^3$These are the target parameters for the adaptive optics system on Mount Stromlo Observatory, Australian National University, Canberra, Australia.
where $l_{\text{pix}}$ is the size of the photosensor’s pixel, and $f$ is the focal length of the lenslet. The reconstructed wavefront (atmospheric disturbance) is proportional to the angles in Eq. 30. Therefore, one needs to simulate the complete acquisition and reconstruction process in order to evaluate the measurement noise values.

B. The procedure of evaluation of a measurement noise

The comprehensive model of the CCD photosensor [9] was used to obtain the corresponding noise value for the case of different noise levels in the signal from the photosensor. We have considered the cases of bright, medium, and dim guidestars. The light noise, namely photon shot noise and photo-response non-uniformity (PRNU), was considered. The equivalent measurement noise was evaluated for the case of 8 and 10 bit photosensor ADC’s in the WFS.

The noise was generated as follows. First, one layer of turbulence with Kolmogorov spectrum was generated and sensed with the model of WFS; no noise was added by the photosensor model at this time. Next, the same wavefront was sensed with the model of a photosensor that added the photon shot noise and the PRNU noise. After each “acquisition” by the model of the photosensor, the numerical model of the ADC performed the quantisation on 8 or 10 bit. Then the two wavefronts were subtracted from each other, and the standard deviation of the remainder signal was calculated. Such a procedure was repeated 32 times in order to obtain an averaged noise level.

C. Results on evaluation of equivalent measurement noise

The results for measurement noise in case of 8 and 10 bit photosensor ADC are summarised in Fig. 4. One can see that the measurement noise levels are similar in both cases.

In the absence of the measurement noise, the residual signal (residual wavefront) is estimated as 46.2 nm in case of 2 frames delay and turbulence standard deviation of 500 nm.

The estimation of the probability density of the measurement noise, which is similar for both the 8 and 10 bit case, is shown in Fig. 5. The measurement noise can be then described as a white Gaussian noise for the purpose of simulations.

These noise parameters were used for the evaluation of the influence of the measurement noise on the discrete LQG controller performance in further Subsection VI.

VI. RESULTS ON SIMULATION OF MEASUREMENT NOISE INFLUENCE ON THE PERFORMANCE OF DISCRETE LQG CONTROLLER

The equivalent measurement noise values were used for the numerical simulations in order to establish the impact of the measurement noise on the residual signal in the LQG control system. The model (see Fig. 1) with the parameters mentioned in Subsection IV-C was used for the simulations. The measurement noise was an additive white Gaussian noise with the value according to the Section V. The cases of 8 and 10 bit ADC’s of the WFS photosensor were considered as the usual choices in the design of modern WFS. The duration of the simulation was set to 3 seconds, and 3000 data points were collected for each run (16 runs were performed for each parameters set).

The residual noise value, which is the standard deviation of the residual signal, are shown in Fig. 6. The estimated probability density for the residual noise is provided in Fig. 7. These results can be interpreted as the remaining uncompensated atmospheric turbulence (output disturbance) that degrades the astronomical images. Ideally, the residual level of signal should be zero; however, even in the absence of measurement noise, the residual signal (residual wavefront) is estimated as 46 nm due to 2 frames
delay in the system. The results in Fig. 6 can be represented using the Strehl ratio from Eq. 13. As above, even in the absence of noise, the residual signal is about 46 nm that gives the Strehl ratio $S = 0.69$. In the presence of measurement noise, the residual signal will increase by the values that are presented in Fig. 4 (these values correspond to the photosensor noise and do not include the noise due to time delays). The LQG controller reduces the noise to the levels, presented in Fig. 6. The results are summarised in Table I in the form of the noise values and corresponding Strehl ratio.

![Fig. 6. The performance of the discrete LQG controller in the presence of the measurement noise: the residual signal (uncompensated atmospheric turbulence) level versus the noise in the photosensor noise.](image)

![Fig. 7. Estimation of the probability density function of the residual noise for the case of CCD photosensor with 10 bit ADC, photon shot noise and PRNU 5% (worst case scenario).](image)

The results summarised in Table I show that the LQG controller attenuates considerably the measurement noise even in the case of relatively strong noise (photon shot noise and 5% PRNU noise).

### VII. Conclusions

In this paper, the problem of adequate modelling of measurement noise and its impact on the performance of the LQG controller for the adaptive optics system was considered. The correspondence between the noise in the Shack-Hartmann wavefront sensor and the measurement noise was established using the comprehensive model of the CCD photosensor. Using the realistic model of the adaptive optics components, the performance of the formulated LQG controller was estimated for the case of dim, medium and bright guidestars.

Some results were predictable, the others were not. For example, we did not expect the ill-conditioning issues and numerical problems in solution of Riccati equations for the augmented state space formulation. Next, the equivalent noise values for 8 and 10 bit photosensor ADC were very close (as expected), but the impact of the PRNU noise was expected to be more significant. Furthermore, the performance of the LQG controller in case of dim guidestar (the case of strong measurement noise) was expected to be worse. This can be explained by the fact that we have used the white Gaussian noise both for the disturbance and the measurement noise.

The numerical simulations show promising results of the performance of the discrete LQG controller even in case of a strong measurement noise and dim guidestars. The actuators coupling, the numerical stability, and more accurate models of the deformable mirror are the subjects of ongoing research.

### REFERENCES


